Bound states in nonrelativistic four-fermion interaction model. [1]

A.V.Sinitskaya, E.V.Pavlova.
Department of Theoretical Physics, Irkutsk State University.
Irkutsk, 664003, RUSSIA.
e-mail: VALL@physdep.irkutsk.su

Abstract

The bound states of two particles are studied in frames of non-relativistic quantum field model with current $\times$ current type interaction by analyzing the Bethe-Salpeter amplitudes. The Bethe-Salpeter equations are obtained in closed form. The existence of Goldstone mode corresponding to the spontaneous breaking of additional SU(2) symmetry of the model is revealed.

The conventional approach to the investigation of two particle bound states in quantum field theory is based on obtaining the Bethe-Salpeter equation (approximate in general) [1]. In the paper [2] for the nonrelativistic model with current $\times$ current type interaction two fermion bound states were examined by straightforward solving of the two particle eigenstate problem for the total Hamiltonian. Since the Heisenberg fields of this model as in relativistic case contain both creation and annihilation operators it seems instructive to follow up the obtaining of the Bethe-Salpeter equation in the frames of this model.

In present paper it is shown that the existence of the closed Bethe-Salpeter equation in this case is related with the fact that Hamiltonian does not contain the "fluctuation" terms [3], i.e. the terms which do not commute with the particle number operator.

Hamiltonian of the model has a form:

$$H = \int d^3x \left[\Psi^{\dagger a}_\alpha(x) \varepsilon(\hat{\vec{p}}) \Psi^a_\alpha(x) - \lambda J^\mu(x) J_\mu(x) \right], \tag{1}$$

where $x = (\vec{x}, t)$,

$$\begin{aligned} &\varepsilon(\hat{\vec{p}}) e^{i\vec{k}\vec{x}} = \varepsilon(\vec{k}) e^{i\vec{k}\vec{x}}, \;\; \hat{\vec{p}} = -i\vec{\nabla}, \;\; J^0 = \Psi^{\dagger a}_\alpha(x)\Psi^a_\alpha(x), \\ &\vec{J}(x) = \frac{1}{2mc}\left(\Psi^{\dagger a}_\alpha(x)\, \hat{\vec{p}} \Psi^a_\alpha(x) - \hat{\vec{p}} \Psi^{\dagger a}_\alpha(x) \Psi^a_\alpha(x) \right), \end{aligned} \tag{2}$$

$\alpha = 1, 2$ is isospin index, $a = 1, 2$ is an index of the additional degrees of freedom. $\varepsilon(\vec{k}) = \frac{\vec{k}^2}{2m} + mc^2$ - is a "bare" fermion spectrum. $\Psi^a_\alpha(x)$ is a Heisenberg fermionic field which at $t = 0$ has the form:

$$\Psi^a_\alpha(\vec{x}, 0) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \{ e^{i\vec{k}\vec{x}} f^a(\vec{k}) A_\alpha(\vec{k}) + e^{-i\vec{k}\vec{x}} g^a(\vec{k}) \tilde{A}^\dagger_\alpha(\vec{k}) \}, \tag{3}$$

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where $A_\alpha(\vec{k}),\ \ \tilde{A}_\alpha(\vec{k})$ are annihilation operators of two different kinds of fermions. They satisfy the canonical anticommutation relations:

$$\left\{A_\alpha(\vec{k}),\ A^\dagger_\beta(\vec{q})\right\} = \left\{\tilde{A}_\alpha(\vec{k}),\ \tilde{A}^\dagger_\beta(\vec{q})\right\} = \delta_{\alpha\beta}\delta^3(\vec{k}-\vec{q}), \tag{4}$$

The vacuum state is defined with respect to both kinds of particles $A_\alpha(\vec{k})$, $\tilde{A}_\alpha(\vec{k})$:

$$A_\alpha \mid 0\rangle_{A\tilde{A}} = \tilde{A}_\alpha \mid 0\rangle_{A\tilde{A}} = 0 \tag{5}$$

The amplitudes $f^a(\vec{k}),\ g^a(\vec{k})$ satisfy the completeness condition:

$$f^a(\vec{k})\bar{f}^b(\vec{k}) + g^a(-\vec{k})\bar{g}^b(-\vec{k}) = \delta^{ab}; \quad a,b = 1,2. \tag{6}$$

It was shown at [4] that the existence of the exact solution for the Hamiltonian implies the condition $f^a,\ g^a = const$. At this condition Hamiltonian does not depend on $f^a,\ g^a$, what allows to take them in the following form:

$$f^a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \delta_{a1}, \quad g^a = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \delta_{a2}. \tag{7}$$

So the field $\Psi^a_\alpha(x)$ can be presented via "frequency" parts $\Psi^{(-)}_\alpha(x)$ and $\Psi^{(+)}_\alpha(x)$:

$$\Psi^a_\alpha(x) = f^a\Psi^{(-)}_\alpha(x) + g^a\Psi^{(+)}_\alpha(x), \tag{8}$$

where

$$\begin{aligned}\Psi^{(-)}_\alpha(\vec{x},0) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k e^{i\vec{k}\vec{x}}A_\alpha(\vec{k}) \\ \Psi^{(+)}_\alpha(\vec{x},0) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k e^{-i\vec{k}\vec{x}}\tilde{A}^\dagger_\alpha(\vec{k}).\end{aligned} \tag{9}$$

Let us write the Heisenberg equations for "frequency parts":

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi^{(-)}_\alpha(x) = &\quad [\Psi^{(-)}_\alpha(x), H] = E_A\Psi^{(-)}_\alpha(x) + V^\dagger_\alpha(x), \\ -i\frac{\partial}{\partial t}\Psi^{(+)}_\alpha(x) = & - [\Psi^{(+)}_\alpha(x), H] = E_{\tilde{A}}\Psi^{(+)}_\alpha(x) + \tilde{V}^\dagger_\alpha(x). \end{aligned} \tag{10}$$

Because of the nonrenormalizability of the model under the consideration one should introduce an ultraviolet cut-off Λ:

$$\frac{1}{V^*} \equiv \frac{1}{(2\pi)^3}\int^\Lambda d^3k = \frac{1}{6\pi^2}\Lambda^3; \quad < k^2 > \equiv \frac{\int^\Lambda \vec{k}^2 d^3k}{\int^\Lambda d^3k} = \frac{3}{5}\Lambda^2; \quad g \equiv \frac{\lambda}{V^*}. \tag{11}$$

Renormalized coupling constant g has dimension of energy and enters alone into the final expressions for the all dynamical characteristics.

There are two point of views on the value of cut-off parameter Λ. First one is to remain Λ finite choosing it by physical sense ([5],[6]). The second point of view is to consider Λ

as a regularization parameter only and tend it to infinity in final expressions, supposing the definite behavior over Λ of "bare" quantities ([7]).

Direct calculations in (10) lead to the expressions for one particle energies:

$$
\begin{aligned}
E_A(\vec{k}) &= \frac{k^2}{2m_A} + mc^2 - 5g + g\frac{< k^2 >}{4m^2c^2}; \\
E_{\tilde{A}}(\vec{k}) &= \frac{k^2}{2m_{\tilde{A}}} - mc^2 + 3g + g\frac{< k^2 >}{4m^2c^2}; \\
m_A &= \frac{m}{\frac{g}{2mc^2}+1}; \quad m_{\tilde{A}} = \frac{m}{\frac{g}{2mc^2}-1}; \quad m = \frac{m_A}{2}\left(1+\sqrt{1+\frac{2g}{m_Ac^2}}\right).
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
V^\dagger_\alpha(x) &= -2\lambda\Psi^{\dagger(+)}_\gamma(x)\Psi^{(-)}_\gamma(x)\Psi^{(-)}_\alpha(x) + 2\lambda\Psi^{(+)}_\gamma(x)\Psi^{\dagger(-)}_\gamma(x)\Psi^{(-)}_\alpha(x) + \\
&+\frac{4\lambda}{(2mc)^2}\Psi^{\dagger(+)}_\gamma(x)\hat{\vec{p}}\Psi^{(-)}_\gamma(x)\hat{\vec{p}}\Psi^{(-)}_\alpha(x) - \frac{4\lambda}{(2mc)^2}\hat{\vec{p}}\Psi^{\dagger(+)}_\gamma(x)\Psi^{(-)}_\gamma(x)\hat{\vec{p}}\Psi^{(-)}_\alpha(x) + \\
&+\frac{4\lambda}{(2mc)^2}\Psi^{(+)}_\gamma(x)\hat{\vec{p}}\Psi^{\dagger(-)}_\gamma(x)\hat{\vec{p}}\Psi^{(-)}_\alpha(x) - \frac{4\lambda}{(2mc)^2}\hat{\vec{p}}\Psi^{(+)}_\gamma(x)\Psi^{\dagger(-)}_\gamma(x)\hat{\vec{p}}\Psi^{(-)}_\alpha(x) + \\
&+\frac{2\lambda}{(2mc)^2}\hat{\vec{p}}\left(\Psi^{\dagger(+)}_\gamma(x)\hat{\vec{p}}\Psi^{(-)}_\gamma(x)\right)\Psi^{(-)}_\alpha(x) - \frac{2\lambda}{(2mc)^2}\hat{\vec{p}}\left(\hat{\vec{p}}\Psi^{\dagger(+)}_\gamma(x)\Psi^{(-)}_\gamma(x)\right)\Psi^{(-)}_\alpha(x) + \\
&+\frac{2\lambda}{(2mc)^2}\hat{\vec{p}}\left(\Psi^{(+)}_\gamma(x)\hat{\vec{p}}\Psi^{\dagger(-)}_\gamma(x)\right)\Psi^{(-)}_\alpha(x) - \frac{2\lambda}{(2mc)^2}\hat{\vec{p}}\left(\hat{\vec{p}}\Psi^{(+)}_\gamma(x)\Psi^{\dagger(-)}_\gamma(x)\right)\Psi^{(-)}_\alpha(x), \\
\tilde{V}^\dagger_\alpha(x) &= 2\lambda\Psi^{\dagger(+)}_\gamma(x)\Psi^{(-)}_\gamma(x)\Psi^{\dagger(-)}_\alpha(x) - 2\lambda\Psi^{(+)}_\gamma(x)\Psi^{\dagger(-)}_\gamma(x)\Psi^{\dagger(-)}_\alpha(x) + \\
&+\frac{4\lambda}{(2mc)^2}\Psi^{\dagger(+)}_\gamma(x)\hat{\vec{p}}\Psi^{(-)}_\gamma(x)\hat{\vec{p}}\Psi^{\dagger(-)}_\alpha(x) - \frac{4\lambda}{(2mc)^2}\hat{\vec{p}}\Psi^{\dagger(+)}_\gamma(x)\Psi^{(-)}_\gamma(x)\hat{\vec{p}}\Psi^{\dagger(-)}_\alpha(x) + \\
&+\frac{4\lambda}{(2mc)^2}\Psi^{(+)}_\gamma(x)\hat{\vec{p}}\Psi^{\dagger(-)}_\gamma(x)\hat{\vec{p}}\Psi^{\dagger(-)}_\alpha(x) - \frac{4\lambda}{(2mc)^2}\hat{\vec{p}}\Psi^{(+)}_\gamma(x)\Psi^{\dagger(-)}_\gamma(x)\hat{\vec{p}}\Psi^{\dagger(-)}_\alpha(x) + \\
&+\frac{2\lambda}{(2mc)^2}\hat{\vec{p}}\left(\Psi^{\dagger(+)}_\gamma(x)\hat{\vec{p}}\Psi^{(-)}_\gamma(x)\right)\Psi^{\dagger(-)}_\alpha(x) - \frac{2\lambda}{(2mc)^2}\hat{\vec{p}}\left(\hat{\vec{p}}\Psi^{\dagger(+)}_\gamma(x)\Psi^{(-)}_\gamma(x)\right)\Psi^{\dagger(-)}_\alpha(x) + \\
&+\frac{2\lambda}{(2mc)^2}\hat{\vec{p}}\left(\Psi^{(+)}_\gamma(x)\hat{\vec{p}}\Psi^{\dagger(-)}_\gamma(x)\right)\Psi^{\dagger(-)}_\alpha(x) - \frac{2\lambda}{(2mc)^2}\hat{\vec{p}}\left(\hat{\vec{p}}\Psi^{(+)}_\gamma(x)\Psi^{\dagger(-)}_\gamma(x)\right)\Psi^{\dagger(-)}_\alpha(x).
\end{aligned}
\tag{13}
$$

As one can see threelinear terms $V^\dagger_\alpha, \tilde{V}^\dagger_\alpha$ acting on vacuum vanish

$$
V^\dagger_\alpha \mid 0\rangle = \tilde{V}^\dagger_\alpha \mid 0\rangle = \langle 0 \mid V^\dagger_\alpha = \langle 0 \mid \tilde{V}^\dagger_\alpha = 0, \tag{14}
$$

due to the absence of "fluctuational terms" in the Hamiltonian (1). This fact turns out to be essential condition for Bethe-Salpeter equation to have closed form.

Let us consider the Bethe-Salpeter (BS) amplitudes for fermionic fields $\Psi^a_\alpha(x)$:

$$
\begin{aligned}
G^{AA,ab}_{\alpha\beta}(x,y;\vec{P}) &= \langle 0 \mid T\left(\Psi^a_\alpha(x)\Psi^b_\beta(y)\right) \mid D_{AA}(\vec{P})\rangle, \\
G^{\tilde{A}\tilde{A},ab}_{\alpha\beta}(x,y;\vec{P}) &= \langle 0 \mid T\left(\Psi^{\dagger a}_\alpha(x)\Psi^{\dagger b}_\beta(y)\right) \mid D_{\tilde{A}\tilde{A}}(\vec{P})\rangle, \\
G^{A\tilde{A},ab}_{\alpha\beta}(x,y;\vec{P}) &= \langle 0 \mid T\left(\Psi^a_\alpha(x)\Psi^{\dagger b}_\beta(y)\right) \mid D_{A\tilde{A}}(\vec{P})\rangle,
\end{aligned}
\tag{15}
$$

where $\mid D_{AA}(\vec{P})\rangle$, $\mid D_{\tilde{A}\tilde{A}}(\vec{P})\rangle$ and $\mid D_{A\tilde{A}}(\vec{P})\rangle$ are two particle bound states of three possible kinds: $A-A$, $\tilde{A}-\tilde{A}$, $A-\tilde{A}$ particles respectively. The action of Hamiltonian on these states gives the equation for bound states energies $\mu_{AA}(\vec{P})$, $\mu_{\tilde{A}\tilde{A}}(\vec{P})$, $\mu_{A\tilde{A}}(\vec{P})$:

$$
\begin{aligned}
H \mid D_{AA}(\vec{P})\rangle &= (W_0 + \mu_{AA}(\vec{P})) \mid D_{AA}(\vec{P})\rangle , \\
H \mid D_{\tilde{A}\tilde{A}}(\vec{P})\rangle &= (W_0 + \mu_{\tilde{A}\tilde{A}}(\vec{P})) \mid D_{\tilde{A}\tilde{A}}(\vec{P})\rangle , \\
H \mid D_{A\tilde{A}}(\vec{P})\rangle &= (W_0 + \mu_{A\tilde{A}}(\vec{P})) \mid D_{A\tilde{A}}(\vec{P})\rangle .
\end{aligned} \tag{16}
$$

W_0 is the vacuum energy.

Let us rewrite the BS amplitudes via "frequency" parts. One can see that matrix elements containing creation operators vanish. Therefore, we have:

$$
\begin{aligned}
G^{AA,ab}_{\alpha\beta}(x,y;\vec{P}) &= f^a f^b \langle 0 \mid T\left(\Psi^{(-)}_\alpha(x)\Psi^{(-)}_\beta(y)\right) \mid D_{AA}(\vec{P})\rangle , \\
G^{\tilde{A}\tilde{A},ab}_{\alpha\beta}(x,y;\vec{P}) &= \bar{g}^a \bar{g}^b \langle 0 \mid T\left(\Psi^{\dagger(-)}_\alpha(x)\Psi^{\dagger(-)}_\beta(y)\right) \mid D_{\tilde{A}\tilde{A}}(\vec{P})\rangle , \\
G^{A\tilde{A},ab}_{\alpha\beta}(x,y;\vec{P}) &= f^a \bar{g}^b \langle 0 \mid T\left(\Psi^{(-)}_\alpha(x)\Psi^{\dagger(-)}_\beta(y)\right) \mid D_{A\tilde{A}}(\vec{P})\rangle .
\end{aligned} \tag{17}
$$

So instead of BS amplitudes (15) we will consider the ones written via "frequency" parts which carry the isospin indexes only:

$$
\begin{aligned}
G^{AA,ab}_{\alpha\beta}(x,y;\vec{P}) &= f^a f^b G^{AA}_{\alpha\beta}(x,y;\vec{P}), \\
G^{\tilde{A}\tilde{A},ab}_{\alpha\beta}(x,y;\vec{P}) &= \bar{g}^a \bar{g}^b G^{\tilde{A}\tilde{A}}_{\alpha\beta}(x,y;\vec{P}), \\
G^{A\tilde{A},ab}_{\alpha\beta}(x,y;\vec{P}) &= f^a \bar{g}^b G^{A\tilde{A}}_{\alpha\beta}(x,y;\vec{P}).
\end{aligned} \tag{18}
$$

Thus using (10) we derive differential equations on these BS amplitudes. Let us write for example the one for $G^{AA}_{\alpha\beta}(x,y;\vec{P})$:

$$
\begin{aligned}
&\left(i\frac{\partial}{\partial t_x} - E_A(\nabla_x)\right) G^{AA}_{\alpha\beta}(x,y;\vec{P}) = \langle 0 \mid T\left(V^\dagger_\alpha(x)\Psi^{(-)}_\beta(y)\right) \mid D_{AA}(\vec{P})\rangle , \\
&\left(i\frac{\partial}{\partial t_y} - E_A(\nabla_y)\right)\left(i\frac{\partial}{\partial t_x} - E_A(\nabla_x)\right) G^{AA}_{\alpha\beta}(x,y;\vec{P}) = \\
&= -i\delta(t_x - t_y)\left\langle 0 \mid \left\{V^\dagger_\alpha(x), \Psi^{(-)}_\beta(y)\right\} \mid D_{AA}(\vec{P})\right\rangle = \\
&= -2i\lambda\left\{1 - \frac{1}{(2mc)^2}\left[\hat{\vec{p}}^2_\xi - \hat{\vec{p}}^2_x + 2\hat{\vec{p}}_\eta(\hat{\vec{p}}_\xi - \hat{\vec{p}}_x)\right]\right\}\delta^4(x-y)G^{AA}_{\alpha\beta}(\eta,\xi;\vec{P})|_{x=\xi=\eta}.
\end{aligned} \tag{19}
$$

Here it is taken into account that at equal times

$$
\langle 0 \mid T\left(\Psi^{(-)}_\alpha(x)\Psi^{(-)}_\beta(y)\right) \mid D_{AA}(\vec{P})\rangle|_{t_x=t_y=t} = \langle 0 \mid \Psi^{(-)}_\alpha(\vec{x},t)\Psi^{(-)}_\beta(\vec{y},t) \mid D_{AA}(\vec{P})\rangle|_{t_x=t_y=t}.
$$

The corresponding expressions for $G^{\tilde{A}\tilde{A}}_{\alpha\beta}(x,y;\vec{P})$ and $G^{A\tilde{A}}_{\alpha\beta}(x,y;\vec{P})$ could be obtained by the same way. The first of them has the similar form as (19), and the last one differs by sign of contribution from zero component $J^0(x)$ in the Hamiltonian (1).

The expression (19) can be rewritten in the integral form:

$$G^{AA}_{\alpha\beta}(x,y;\vec{P}) = 2i\lambda \int d^4\sigma d^4 z \bigtriangleup (x-z) \bigtriangleup (y-\sigma) \{1- \tag{20}$$
$$- \frac{1}{(2mc)^2}\left[\hat{\vec{p}}^2_\xi - \hat{\vec{p}}^2_x + 2\hat{\vec{p}}_\eta(\hat{\vec{p}}_\xi - \hat{\vec{p}}_x)\right]\right\} \delta^4(z-\sigma) G^{AA}_{\alpha\beta}(\eta,\xi;\vec{P})|_{z=\xi=\eta},$$

where $\triangle(x)$ is the casual (coinciding with retarded) Green function, satisfying the equation:

$$\left(i\frac{\partial}{\partial t_x} - E_A(\nabla_x)\right) \bigtriangleup (x-z) = i\delta^4(x-z). \tag{21}$$

Putting now $t_x = t_y = t$ we obtain the equation for instantaneous BS amplitude:

$$G^{AA}_{\alpha\beta}(\vec{x},\vec{y},t;\vec{P}) = 2i\lambda \int d^4 z \bigtriangleup (x-z) \{1- \tag{22}$$
$$- \frac{1}{(2mc)^2}\left[\hat{\vec{p}}^2_\xi - \hat{\vec{p}}^2_x + 2\hat{\vec{p}}_\eta(\hat{\vec{p}}_\xi - \hat{\vec{p}}_x)\right]\right\} \bigtriangleup (y-z) G^{AA}_{\alpha\beta}(\eta,\xi;\vec{P})|_{z=\xi=\eta}.$$

Let us pass to new variables $\vec{R} = \frac{1}{2}(\vec{x}+\vec{y})$, $\vec{r} = \vec{x}-\vec{y}$ - total and relative coordinates respectively. Then BS amplitude has the form:

$$G^{AA}_{\alpha\beta}(\vec{x},\vec{y},t;\vec{P}) = e^{-i\mu_{AA}(\vec{P})t + i\vec{P}\vec{R}} K^{AA}_{\alpha\beta}(\vec{r},0;\vec{P}), \tag{23}$$

where

$$K^{AA}_{\alpha\beta}(\vec{r},0;\vec{P}) = \langle 0 \mid \Psi^{(-)}_\alpha(\frac{\vec{r}}{2},0)\Psi^{(-)}_\beta(\frac{-\vec{r}}{2},0) \mid D_{AA}(\vec{P})\rangle. \tag{24}$$

Combaining all mentioned above and passing in the eq (22) to the momentum representation (the expressions for another two cases are obtained by the same way) we have:

$$F^{AA,\tilde{A}\tilde{A}}_{\alpha\beta}(\vec{s};\vec{P}) = \lambda\frac{2}{(2\pi)^3}\int d^3k \frac{1+(2mc)^{-2}[(\vec{k}+\vec{s})^2 - \vec{P}^2]}{E_{A,\tilde{A}}(\frac{\vec{P}}{2}+\vec{k}) + E_{A,\tilde{A}}(\frac{\vec{P}}{2}-\vec{k}) - \mu_{AA,\tilde{A}\tilde{A}}(\vec{P})} F^{AA,\tilde{A}\tilde{A}}_{\alpha\beta}(\vec{k};\vec{P}), \tag{25}$$

$$F^{A\tilde{A}}_{\alpha\beta}(\vec{s};\vec{P}) = \lambda\frac{2}{(2\pi)^3}\int d^3k \frac{-1+(2mc)^{-2}[(\vec{k}+\vec{s})^2 - \vec{P}^2]}{E_{A}(\frac{\vec{P}}{2}+\vec{k}) + E_{\tilde{A}}(\frac{\vec{P}}{2}-\vec{k}) - \mu_{A\tilde{A}}(\vec{P})} F^{A\tilde{A}}_{\alpha\beta}(\vec{k};\vec{P}). \tag{26}$$

This is exactly the equations obtained in frameworks of eigenstate problem. The functions

$$D^{AA}_{\alpha\beta}(\vec{s};\vec{P}) = \frac{F^{AA}_{\alpha\beta}(\vec{s};\vec{P})}{\left(E_A(\frac{\vec{P}}{2}+\vec{s}) + E_A(\frac{\vec{P}}{2}-\vec{s}) - \mu_{AA}(\vec{P})\right)}, \tag{27}$$
$$D^{\tilde{A}\tilde{A}}_{\alpha\beta}(\vec{s};\vec{P}) = \frac{F^{\tilde{A}\tilde{A}}_{\alpha\beta}(\vec{s};\vec{P})}{\left(E_{\tilde{A}}(\frac{\vec{P}}{2}+\vec{s}) + E_{\tilde{A}}(\frac{\vec{P}}{2}-\vec{s}) - \mu_{\tilde{A}\tilde{A}}(\vec{P})\right)},$$
$$D^{A\tilde{A}}_{\alpha\beta}(\vec{s};\vec{P}) = \frac{F^{A\tilde{A}}_{\alpha\beta}(\vec{s};\vec{P})}{\left(E_A(\frac{\vec{P}}{2}+\vec{s}) + E_{\tilde{A}}(\frac{\vec{P}}{2}-\vec{s}) - \mu_{A\tilde{A}}(\vec{P})\right)},$$

have the meaning of bound states wave functions.

So we have got the closed equations for $D^{AA}_{\alpha\beta}(\vec{k};\vec{P})$, $D^{\tilde{A}\tilde{A}}_{\alpha\beta}(\vec{k};\vec{P})$ and $D^{A\tilde{A}}_{\alpha\beta}(\vec{k};\vec{P})$. The solutions for (25,26) would have the following form:

$$F_{\alpha\beta} = A_{\alpha\beta}(\vec{P}) + \vec{k}^2 B_{\alpha\beta}(\vec{P}) + \vec{k}\vec{C}_{\alpha\beta}(\vec{P}). \tag{28}$$

Here $A_{\alpha\beta}(\vec{P})$, $B_{\alpha\beta}(\vec{P})$, $\vec{C}_{\alpha\beta}(\vec{P})$ are matrixes depending only on $\vec{P}$.

I). First, let us consider the equations (25) for the bound states of the same kind particles, analyzing the bound state rest-frame case, that is $\vec{P}=0$. As one can see, the wave functions for these bound states have a definite symmetry:

$$D^{AA}_{\alpha\beta}(\vec{k};0) = -D^{AA}_{\beta\alpha}(-\vec{k};0), \quad D^{\tilde{A}\tilde{A}}_{\alpha\beta}(\vec{k};0) = -D^{\tilde{A}\tilde{A}}_{\beta\alpha}(-\vec{k};0). \tag{29}$$

Therefore symmetrical and skewsymmetrical parts of $D^{AA}(\vec{k};0)$, $D^{\tilde{A}\tilde{A}}(\vec{k};0)$ are splitted. Then in solution (28) ($A_{\alpha\beta}, B_{\alpha\beta}, \vec{C}_{\alpha\beta}$ are constant matrices) skewsymmetrical $A_{\alpha\beta}$ and $B_{\alpha\beta}$ and symmetrical $\vec{C}_{\alpha\beta}$ over α, β matrices contribute independently to the bound states and correspond to isoscalar and isovector states. Hence it follows, that $A_{\alpha\beta} = A\epsilon_{\alpha\beta}$, $B_{\alpha\beta} = B\epsilon_{\alpha\beta}$ and $\vec{C}_{\alpha\beta}$ can be expanded over three symmetrical matrices: I, τ_1, τ_3. According to these remarks the equation (25) is brought to the following set of equations:

$$\begin{aligned} A &= I_0 A + (2mc)^{-2} I_1 A + I_1 B + (2mc)^{-2} I_2 B, \\ B &= (2mc)^{-2} I_0 A + (2mc)^{-2} I_1 B, \\ C^i_{\alpha\beta} &= (2mc)^{-2} I^{ij} C^j_{\alpha\beta}. \end{aligned} \tag{30}$$

Here

$$I_n = \frac{2\lambda}{(2\pi)^3}\int d^3k \frac{(\vec{k}^2)^n}{2E_A(\vec{k}) - \mu_s}; \quad I^{ij} = \frac{2\lambda}{(2\pi)^3}\int d^3k \frac{2\vec{k}^i\vec{k}^j}{2E_A(\vec{k}) - \mu_v}. \tag{31}$$

μ_s, $\mu_v \equiv \mu_{AA}(0)$, $\mu_{\tilde{A}\tilde{A}}$ stand for isoscalar and isovector states masses of bound states AA, $\tilde{A}\tilde{A}$ accordingly. Let us introduce the dimensionless parameter $G = 2g/(m_A c^2)$ and the parameters:

$$\chi_s^2 = m_A(2m_A c^2 - \mu_s), \quad \chi_v^2 = m_A(2m_A c^2 - \mu_v). \tag{32}$$

The equations on these parameters χ_s, χ_v follow from (30).

From the last relation of (30) using (31) we obtain usual "gap" equation for isovector state:

$$1 = \frac{\lambda}{(2\pi)^3}\frac{m_A}{3m^2c^2}\int^{\Lambda} d^3k \frac{\vec{k}^2}{\vec{k}^2 + \chi_v^2} \tag{33}$$

First two equations (30) form linear homogeneous system with respect to A and B. Demanding the determinant of this system to be zero we come to the equation on μ_s:

$$\left(I_1 - (2mc)^2\right)^2 = I_0\left(I_2 + (2mc)^4\right). \tag{34}$$

Then from (34) the equation on χ_s^2 follows:

$$\frac{\lambda m_A}{(2\pi)^3}\int^{\Lambda}\frac{d^3k}{\vec{k}^2+\chi^2}=\left(\frac{m_A g}{(2mc)^2}-\frac{1}{2}\right)^2\left[\frac{1}{2}-\frac{\chi^2}{(2mc)^2}+\frac{m_A g}{(2mc)^2}\cdot\frac{<\vec{k}^2>+\chi^2}{(2mc)^2}\right]^{-1} \tag{35}$$

As it was shown in [4] there are three inequivalent operator realizations of the Hamiltonian (1) called "A", "B" and "C" cases. The demand in "B" case corresponding to zero vacuum energy for the one particle energy to have the usual nonrelativistic form, $E_B(\vec{k})=(2m_A)^{-1}k^2+m_Ac^2$, leads to the expressions: $<k^2>=m_A^2c^2\left[1+G+\sqrt{1+G}\right]$, allowing to exclude the parameter $<k^2>$. Then after integration in the l.h.s. of equation (35) we derive transcendental equation:

$$(z^2c_1-c_2)(z-\arctan z)=z^3, \tag{36}$$

where

$$c_1=\frac{9}{20}\cdot\frac{3+2G+\sqrt{1+G}}{1+G+\sqrt{1+G}},\quad c_2=\frac{3}{4}G\Big(1+\frac{2}{1+\sqrt{1+G}}\Big),\quad z=\frac{\Lambda}{\chi_s}$$

The analysis of this equation shows that it always has a solution at $c_1>1$. With a good accuracy $c_1\simeq\frac{9}{10}G$, therefore, the condition of the existence of a solution for bound state with zero isospin reduces to $G>\frac{10}{9}$. From the numerical solution of the equation (36) follows that $z(G)$ strongly changes in the region $\frac{10}{9}<G\leq 1.3$, but further on, at $G\geq 1.3$, is slowly achieving its asymptotic value $z(\infty)=\sqrt{\frac{5}{6}}$.

II).Now let us consider the equation (26) for wave function of the two different kind particle bound state. In this case there is no such a symmetry as took place for bound states of particles AA and $\tilde{A}\tilde{A}$ (29). Therefore the solution (28) can not be splitted, and one should solve the homogeneous system of three equations for matrixes $A_{\alpha\beta}(\vec{P})$, $B_{\alpha\beta}(\vec{P})$, $C^i_{\alpha\beta}(\vec{P})$. The denominator of (26) contains two different one-particle excitations spectra $E_A(\frac{\vec{P}}{2}+\vec{k})$ and $E_{\tilde{A}}(\frac{\vec{P}}{2}-\vec{k})$. Using the explicit form of these spectra (12) and changing the variable $\vec{k}\ \rightarrow\ \vec{\kappa}\ =\ \vec{k}\ +\ \vec{P}\gamma$ where $\gamma\ =\ mc^2/g\ =\ (1+\sqrt{1+G})/G$ one could obtain the same (symmetrical over $\vec{\kappa}$) structure of the denominator in the kernel of the equation (26) as in previous case. The system of equations to determine the matrixes $A_{\alpha\beta}(\vec{P})$, $B_{\alpha\beta}(\vec{P})$, $C^i_{\alpha\beta}(\vec{P})$ now reads:

$$\begin{aligned}
&\Big(-(2mc)^2I_0(\vec{P})+\vec{P}^2((2\gamma)^2-1)I_0(\vec{P})+I_1(\vec{P})-1\Big)\,A_{\alpha\beta}(\vec{P})+\\
&+\Big(-(2mc)^2+\vec{P}^2((2\gamma)^2-1)I_1(\vec{P})+I_2(\vec{P})\Big)\,B_{\alpha\beta}(\vec{P})-\frac{4}{3}\gamma I_1(\vec{P})\vec{P}\vec{C}_{\alpha\beta}(\vec{P})=0,\\
&I_0(\vec{P})A_{\alpha\beta}(\vec{P})+(I_1(\vec{P})-1)B_{\alpha\beta}(\vec{P})=0,\\
&4\gamma I_0(\vec{P})\vec{P}A_{\alpha\beta}(\vec{P})+4\gamma I_1(\vec{P})\vec{P}B_{\alpha\beta}(\vec{P})-\Big(\frac{2}{3}I_1(\vec{P})-1\Big)\,\vec{C}_{\alpha\beta}(\vec{P})=0,
\end{aligned} \tag{37}$$

$$\text{where}\quad M^2(\vec{P})=\frac{\vec{P}^2}{4}(1-4\gamma^2)-(2mc)^2+<\vec{k}^2>-2m\gamma\mu(\vec{P});$$

$$I_n(\vec{P})=\frac{V^*}{(2\pi)^3}\int\frac{d^3s\,(\vec{s}^{\,2})^n}{\vec{s}^{\,2}+M^2(\vec{P})}, \tag{38}$$

As it is seen from the last equation of (37), the vector $\vec{C}_{\alpha\beta}(\vec{P})$ has the structure $\vec{C}_{\alpha\beta}(\vec{P}) = \vec{P}C_{\alpha\beta}(\vec{P}) + \vec{C}^0_{\alpha\beta}(\vec{P})$. Substituting this expression into the system (37), we obtain at first the equation

$$\frac{2}{3}I_1(\vec{P})\vec{C}^0_{\alpha\beta} = \vec{C}^0_{\alpha\beta}, \quad , \tag{39}$$

and secondly the condition for determinant of the system:

$$\begin{aligned}
&\Big(-(2mc)^2 I_0(\vec{P}) + \vec{P}^2((2\gamma)^2 - 1)I_0(\vec{P}) + I_1(\vec{P}) - 1\Big)\, A_{\alpha\beta}(\vec{P}) + \\
&+ \Big(-(2mc)^2 + \vec{P}^2((2\gamma)^2 - 1)I_1(\vec{P}) + I_2(\vec{P})\Big)\, B_{\alpha\beta}(\vec{P}) - \frac{4}{3}\gamma I_1(\vec{P})\vec{P}^2 C_{\alpha\beta}(\vec{P}) = 0, \\
&I_0(\vec{P})A_{\alpha\beta}(\vec{P}) + (I_1(\vec{P}) - 1)B_{\alpha\beta}(\vec{P}) = 0, \\
&4\gamma I_0(\vec{P})A_{\alpha\beta}(\vec{P}) + 4\gamma I_1(\vec{P})B_{\alpha\beta}(\vec{P}) - \left(\frac{2}{3}I_1(\vec{P}) - 1\right) C_{\alpha\beta}(\vec{P}) = 0
\end{aligned} \tag{40}$$

to be equal to zero. So the equation (39) determines the solution for isovector state:

$$\frac{2}{3(2\pi)^3}\int \frac{d^3s\,\vec{s}^{\,2}}{\vec{s}^{\,2} + M_1^2(\vec{P})} = \frac{1}{V^*}. \tag{41}$$

The system (40) has no fixed tensor structure so the mentioned condition for determinant of the (40) gives the equation for bound state mass $\mu_{A\tilde{A}}(\vec{P})$ of isoscalar and isovector states in the similar form:

$$\mu^s_{A\tilde{A}}(\vec{P}) = \mu^{v,2}_{A\tilde{A}}(\vec{P}) = \frac{G\vec{P}^2}{4m_A}\left[5 + 12\gamma^2 - \frac{24\gamma^2}{1 + 2M_2^2(\vec{P})I_0(\vec{P})}\right]. \tag{42}$$

At infinitesimal $\vec{P}$ bound state energy $\mu_{A\tilde{A}}(\vec{P})$ tends to zero, that corresponds to the Goldstone mode. The existence of these four Goldstone modes was shown algebraically in the paper [4]. Such a state is generated by the operator $Q^\dagger_{\alpha\beta} = \int d^3k A^\dagger_\alpha(\vec{k})\tilde{A}^\dagger_\alpha(-\vec{k})$:

$$Q \mid 0\rangle = \mid D_{A\tilde{A}}(0)\rangle\,. \tag{43}$$

Therefore the wave function $D^{A\tilde{A}}_{\alpha\beta}(\vec{k};0) = 1$ at $\vec{P} = 0$. Let us substitute the solution (28) into eq.(26) at $\vec{P} = 0$, taking into account that $\vec{C}_{\alpha\beta}(0) = 0$. Thus we obtain the relation

$$-2g + g\frac{\vec{k}^2}{2m^2c^2} + g\frac{<\vec{k}^2>}{2m^2c^2} = E_A(\vec{k}) + E_{\tilde{A}}(-\vec{k}), \tag{44}$$

which according to (12) is identity independently of the cut-off Λ value.

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